LINEAR ANALYSIS OF ELECTRONIC SWITCHING*

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ABSTRACT. A linear analysis of the multivibrator circuit has been made and it is shown that the time of switch over between two relaxation states can be calculated with a fair degree of accuracy. Variation of the time of rise with various circuit parameters has been studied and results have been compared with observational data By means of this analysis a criterion for the design of fast switching circuits has been indicated

The current-voltage relationship in a vacuum tube is non-linear and, therefore, any exact analysis of circuits using such tubes would have to depend on the solution of non linear differential equations. In fact, considering the actual grid voltage-plate current characteristic of a tube one may have to resort to graphical methods since analytical expressions for such characteristics are only approximations. It is because of these facts that considerable difficulties have been experienced in the analysis, in particular, of relaxation circuits, which have otherwise proved to be so useful in a very large number of modern timing, switching and controlling devices

In the case of multivibrators, which form a basis of so many above-mentioned circuits, a number of papers have appeared, *e g.* Kiebert and Inglis (1945), on the calculation of the relaxation periods. The switching period, which lies between two successive relaxations, has not been fully investigated (Williams, Aldrich, Woodford, 1950; Ahmed, 1948), and many authors, considering lower frequencies of operation, have simply assumed this period to be zero. When multivibrators or similar other circuits are being used these days in ever faster networks, it becomes imperative to investigate the switching period, even with a small margin of error The designing engineer or physicist must have some quantitative information about the switching time.

The circuit of a symmetrical multivibrator, originally given by Abraham and Bloch (1927), is shown in Fig. 1 and its operation has been explained by several authors. The voltage waveforms at the two grids are shown in Fig 2; it is the period *ab* which is referred to as the switching time, since it is during this time that one tube is being switched on while the other is being switched off. During *ab* both the tubes are conducting and since the output of one is connected to the other, it is during *ab* that the grid voltage of one, due to cumulative amplification, is rapidly rising while

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that of the other is rapidly falling. Basically, therefore, the problem of determining the switch-over time amounts to a solution of the differential equations for the grid voltage in the eqivalent amplifying circuit of Fig. 3 with appropriate boundary conditions.

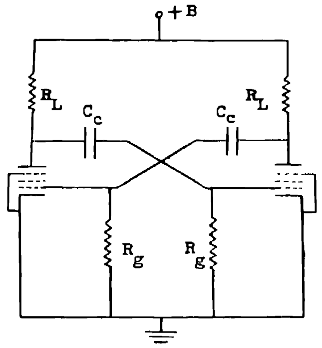


FIG. 1
A symmetrical multivibrator

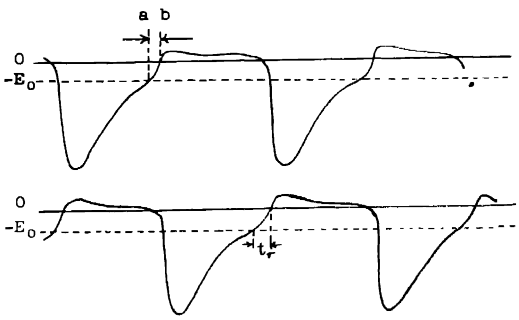


FIG. 2
The grid voltage wave forms of a symmetrical multivibrator

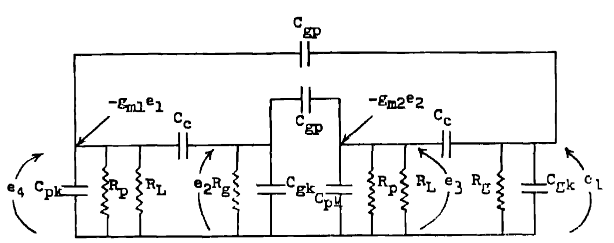


FIG. 3
A-C equivalent circuit of the multivibrator

To facilitate the application of the linear method of circuit analysis some simplifying assumptions have to be made regarding the a-c operation of the multivibrator.

1. It may be assumed that the relaxation time of the circuit is large compared to the switching time. This would permit the omission of the coupling condensers from the equivalent circuit of Fig 3 and is a natural assumption for low frequency multivibrators. When high frequency multivibrators will be considered these coupling condensers will have to be restored.

2. It may be assumed that there is negligible inductance in the components at the frequencies under consideration.

3. It may be assumed that the tubes used have negligible grid to plate capacitance C_{gp}. This assumption would be valid only for pentode tubes, and at lower frequencies.

4. It may also be assumed that during the interval ab the value of the mutual conductance of the two tubes remains constant and is equal to the normal value g_m. In fact, this is the weakest assumption of all, since the full value of the mutual conductance of the tube that starts to be switched on at the instant a (Fig. 2) is not established till somewhat later than a. However, for the sake of linearising the analysis this assumption has to be made, and it comes close to being true, again, in the case of pentode tubes.

With the above-mentioned assumptions the simple equivalent circuit of Fig. 4 is obtained, and its differential equations are given below (p. 285) where

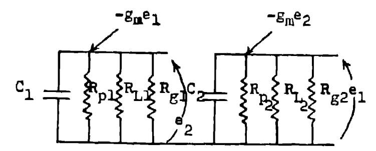


FIG 4

A simplified *a-c* equivalent circuit of the symmetrical multivibrator

the total shunting capacitances and conductances are indicated by the letters C and G, and p denotes the time derivative d/dt.

$$(G_1 + pC_1)e_1 + g_{m2}e_2 = 0 \qquad (1)$$

$$g_{m1}e_1 + (G_2 + pC_2)e_2 = 0 \qquad (2)$$

The determinant of these equations is written in an expanded form below, and it has been equated to zero so that the exponentials arising in the circuit may be determined.

$$p^2C_1C_2 + p(C_1G_2 + C_2G_1) + (G_1G_2 + g_{m1}g_{m2}) = 0 \qquad \cdots \quad (3)$$

If C_1/G_1, the shunt time constant, is denoted by t_{1s}, and C_2/G_2 by t_{2s} the mid-frequency amplification g_{m1}/G_1 is denoted by A_1, and g_{m2}/G_2 by A_2 and $A_1A_2=A^2$, then the roots of (3) are

$$p_{1,2} = -\tfrac{1}{2}(t_{1s}+t_{2s}) \pm \left[\tfrac{1}{4}\left(\frac{1}{t_{1s}} - \frac{1}{t_{2s}}\right)^2 + \frac{A^2}{t_{1s}t_{2s}} \right]^{\frac{1}{2}} \qquad . \quad (4)$$

In a symmetrical circuit the roots will be.

$$p_1 = \frac{(g_m - G)}{C}; \quad p_2 = -\frac{(g_m + G)}{C} \qquad . \quad (5)$$

It is obviously the presence of the positive root p_1 which causes instability and makes the grid voltages e_1 and e_2 rise or fall rapidly to the level where the tube that was previously conducting, now goes off and the other one comes on. It is further seen that the condition of disappearance of the positive root is that A, the mid-frequencp amplification should be less than unity. This result could, of course, be anticipated from a simple feedback consideration, but in this case it only confirms the validity of the calculation It is also clearly indicated that the requirement for a rapid switch-over, or large positive root, is that the quotient g_m/C should be large.

In order to derive an explicit equation for the time of rise for e_1 or e_2 during the switching interval, one of the voltages may be written as follows :

$$e_1 = A_1\epsilon^{p_1t} + A_2\epsilon^{p_2t} \qquad \ldots \quad (6)$$

Before proceeding with the application of the boundary conditions, however, if e_1 is considered to be the expression for the grid voltage of the tube that is being switched on, it is useful to consider the fact that the voltage e_1 starts rising rapidly from a value $-E_0$ (this is not the same as the cut-off voltage $-E_{c\cdot 0}$ in the case of static plate voltage) that is, the grid voltage at which plate current just starts to flow. The switch-over time may, therefore, be defined as the time required by e_1, given in equation (6), to rise from a value zero by an amount $|E_0|$. Alternatively, the expression for e_1 may be modified and put in the form

$$e_1 = A_1\epsilon^{p_1t} + A_2\epsilon^{p_2t} - E_0 \qquad \ldots \quad (7)$$

so that the time of switch-over may be defined as the time required by e_1, given by equation (7), to rise from a value $-E_0$ to zero volts.

It is obvious that these two expressions will yield the same value of the time of switch-over, but it may be added that equation (7) proves to be more convenient for mathematical handling and will therefore be used in all further calculations.

Equation (7 has, then, to be solved with two boundary conditions. The first is that at time $t=0$, $e_1=-E_0$ and therefore $A_1+A_2=0$. The second boundary condition is that at $t=0$, the slope of e_1 given from equation (7) should be the same as the slope of the grid voltage in the preceding relaxation period at t just less than zero. This slope can be proved to be continuous by an elaborate mathematical analysis but the following physical explanation also shows the same. The grid whose voltage e_1 is being considered is the grid of the tube that was off at $t < 0$ and which is coupled through the condenser C_c to the plate of the tube that was conducting current at $t < 0$ and continues to conduct till after $t = 0$. With reference to Fig. 4 g_m is the mutual conductance that is assumed to jump from zero to a constant (normal) value. This jump will cause a discontinuity in the derivative of the current flowing into the plate of this tube, but due to the shunting capacitance the voltage at the plate will be continuous. This voltage is transferred by the coupling condenser to the grid of the second tube, and thence to the plate of the second tube. The quantity $g_{m2}e_2$ is continuous because both its constituents are continuous, and since this current determines the slope de_1/dt, it is concluded that de_1/dt must also be continuous

The relaxing condenser voltage is given by the well-known equation :

$$e_c = (\text{most negative voltage})\ \epsilon^{-t'/\text{ series time constant } T} \quad \ldots \quad (8)$$

where t' is measured from the instant the condenser reaches its most negative voltage, $T = C_c\left(R_g + \dfrac{R_p R_L}{R_p + R_L}\right)$ and $t = 0$ of equation (7) corresponds to that value of t' which makes $e_c = -E_0$. Thus,

$$\left(\frac{de_c}{dt}\right)_{e_c = -E_0} = \left(\frac{de_1}{dt}\right)_{e_1 = -E_0} = \frac{|E_0|}{T} = A_1 p_1 + A_2 p_2$$

and finally,

$$e_1 = \frac{|E_0|}{T(p_1 - p_2)}\left[\epsilon^{p_1 t} - \epsilon^{p_2 t}\right] - E_0 \quad \ldots \quad (9)$$

THE TRANSIENT SWITCH-OVER TIME

It can now be seen that the switch-over time or the time of rise t_r, being the time required for e_1 to become zero, can be easily calculated However, the explicit equation for e_2 can also be derived, and if the time t_{r2}, required by e_2 to go from zero to $-E_0$ is shorter than t_{r1}, then of course the amplifying circuit will be opened at the instant t_{r2} and in such a case t_{r2} will be the effective switch-over time of the whole circuit. It may be noticed that as far as the design work is concerned, the exponentials for e_1 and e^2 being the same, the value of either t_{r1} or t_{r2} can be taken as the basis of the calculation.

In order to have an explicit equation for the time of rise, the exponential term with a negative coefficient may be omitted, because this term rapidly decreases to zero and only affects the form of the grid voltage e_1 in the beginning.

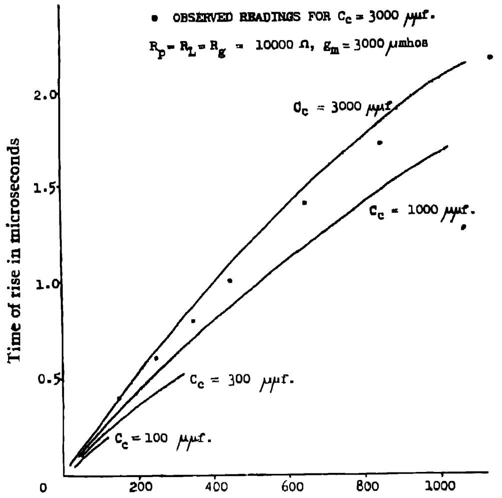
Thus,

$$t_r = \frac{\ln[T(p_1 - p_2)]}{p_1} \qquad \dots (10)$$

$$= \frac{\ln\left[\left(R_g + \frac{R_p R_L}{R_p + R_L}\right) \frac{2 g_m C_c}{C}\right]}{\frac{1}{C}\left(g_m - \frac{R_p R_L + R_g R_L + R_g R_p}{R_p R_g R_L}\right)} \qquad \dots (11)$$

It is interesting to note that the quantity $-E_0$ does not appear in this expression for t_r due to the special boundary conditions.

The calculated variations of the time of rise, when the various elements in the circuit are varied, have been plotted in Figs. 5, 6 and 7 and a set of



Shunting capacities in μμf

FIG. 5

Calculated curves for the variation of the time rise with shunting capacities -g_m constant

observations to check these results has been given in Table I. Further observational results are found in reference (Ahmed, 1948). In connection with plotting the data in Fig. 5, it may be remarked that the observed time of rise for zero external shunt capacitance was taken as the basis for estimating the effective shunting capacitance. This is found to be 150 $\mu\mu f$. (including the effect of C_{gp}) and so 150 $\mu\mu f$ has been added to the externally connected condenser in each observation:

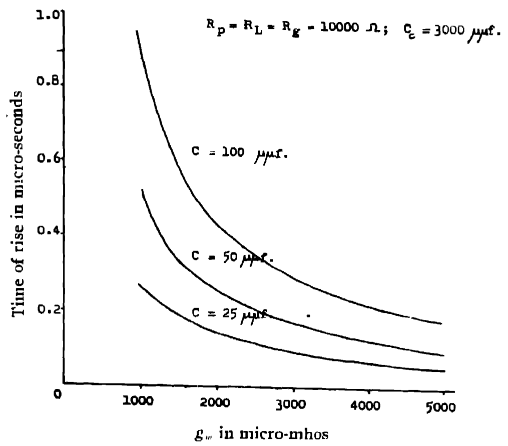


FIG 6

Calculated curves for the variation of the time of rise with mutual conductance C_c constant

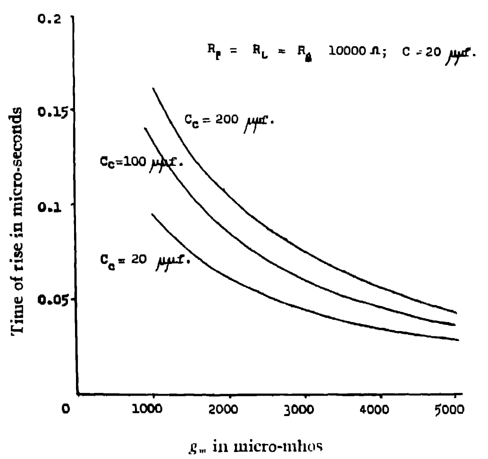


FIG. 7

Calculated curves for the variation of the time of rise with mutual conductence C constant

The divergence between the observed and calculated results is noticeable as the shunting capacitance becomes large as compared to the coupling capacitance. This is natural since the equation for the time of rise was developed on the assumption that the coupling condenser was very much larger than the total shunting capacitance.

TABLE I

Circuit components as in Fig. 5 ; $C_c = 3000\ \mu\mu f$.

External shunting capacitance	Effective shunting capacitance	Switching time
0 $\mu\mu f$	150 $\mu\mu f$.	0.4 μ sec.
100 ,,	250 ,,	0.6 ,,
200 ,,	350 ,,	0.8 ,,
300 ,,	450 ,,	1.0 ,,
500 ,,	650 ,,	1.4 ,,
700 ,,	805 ,,	1.7 ,,
1000 ,,	1150 ,,	2.2 ,,

The effect of grid to plate capacitance C_{gp}

Under the same set of assumptions as above, the exponentials which determine the time of rise are given by :

$$p_1 = \frac{1}{C + 4C_{gp}}(g_m - G) \; ; \quad p_2 = \frac{-1}{C}(g_m + G) \tag{12}$$

These show clearly that the positive root p_1 has been considerably decreased, and that the time of rise will therefore increase when an appreciable amount of grid to plate capacitance is present. When it is desired to reduce the switch-over time in a circuit, it is necessary to minimise the capacitance C_{gp}. A set of observations is given in Table II to illustrate the effect of C_{gp} and a comparison with the previous set will indicate that C_{gp} is several times more effective per micro-farad in changing the time of rise than the shunting capacitance. From these observations, however, it is not possible to separate the C_{gp} from the rest of the shunting capacitance.

TABLE II

Circuit of Fig. 1 with $R_L = R_g = 10\ k\Omega$; $C_c = 3000\ \mu\mu f$. ; 6J5 tubes at 100 volts.

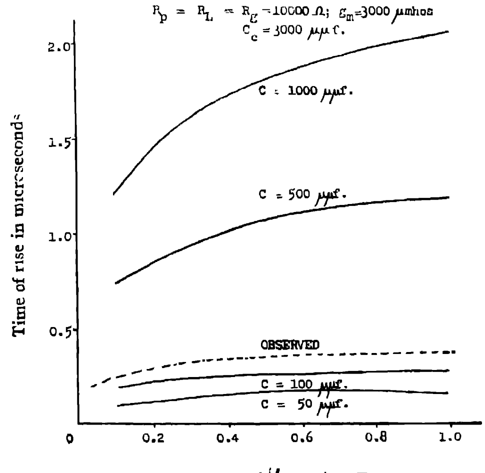
Externally connected C_{gp}	Time of rise
0 $\mu\mu f$	0.4 μ sec.
20 ,,	0.7 ,,
30 ,,	0.9 ,,
50 ,,	1.2 ,,
100 ,,	1.6 ,,
200 ,,	3.0 ,,

The effect of positive return for the grid

The ends of the grid resistances R_g which have so far been connected directly to the cathodes may also be connected to a source of positive potential E_+. This will be a purely d-c condition and will, therefore, not change the differential equations (1) and (2). The discharge of the coupling condenser will, however, now take place under different steady-state conditions and so will affect the time of rise. It may be expected even quantitatively that by making the condenser discharge sharper than before the switch-over time will be shortened. In this case the time of rise may easily be shown to be approximately

$$t_r = \frac{\ln\left[T(p_1 - p_2)\dfrac{|E_0|}{E_+ + |E_0|} \right]}{p_1} \qquad (13)$$

A set of values for the time of rise has been calculated and plotted in Fig. 8 and the observations given in Table III, were taken to check the general form of the result.



Ratio $\lambda = |E_0| / |E_0| + E_+$

FIG. 8
Calculated variations of the time of rise with positive grid return voltage

In conclusion it must be stated that by means of a simple linear analysis of the multivibrator circuit in the state when both tubes conduct, it is possible to obtain a fairly accurate idea of the quantitative value for the time of switch-over. The variations of the time of rise with different circuit components can also be predicted. The difference between the observed and calculated results is explicable on the basis of assumptions made. The equations developed lead to reliable design criteria and, in fact, the author carried out such designs and constructed multivibrators with frequencies of several megacycles per second.

TABLE III

Circuit of Fig. 1 with all resistances = 10 kΩ; C_c = 3000 $\mu\mu$f.; 6J5 tubes with 160 volts on the plates.

Positive return voltage	Peak plate voltage	Cut-off voltage	Time of rise
0 v.	145 v.	−14.0 v.	0.37 μ sec.
25 ,,	112 ,,	−12.1 ,,	0.32 ,,
45 ,,	96 ,,	−11.3 ,,	0.28 ,,
65 ,,	78 ,,	−10.0 ,,	0.26 ,,
85 ,,	60 ,,	−8.0 ,,	0.24 ,,
105 ,,	41 ,,	−6.0 ,,	0.22 ,,

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